

Global existence and estimates of the solutions to nonlinear integral equations

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Abstract

It is proved that a class of nonlinear integral equations of the Volterra-Hammerstein type has a global solution, that is, solutions defined for all $t \geq 0$, and estimates of these solutions as $t \rightarrow \infty$ are obtained. The argument uses a nonlinear differential inequality which was proved by the author and has broad applications.

1 Introduction

Consider the equation:

$$u(t) = \int_0^t e^{-a(t-s)} h(u(s)) ds + f(t) := T(u), \quad t \geq 0; \quad a = \text{const} > 0. \quad (1)$$

that is, Volterra-Hammerstein equation. There is a large literature on nonlinear integral equations, [6], [1]. The usual methods to study such equations include fixed-point theorems such as contraction mapping principle and degree theory, (Schauder and Leray-Schauder theorems). The goal of this paper is to give a new approach to a study of equation (1). We give sufficient conditions for the global existence of solutions to (1) and their estimates as $t \rightarrow \infty$.

Denote $f' := \frac{df}{dt}$. By $c > 0$ various constants will be denoted.

Let us formulate our assumptions:

$$|h(u)| \leq c|u|^b, \quad |h'(u)| \leq c|u|^{b-1}, \quad b \geq 2, \quad (2)$$

$$|f(t)| + a|f'(t)| \leq ce^{-a_1 t}, \quad a_1 = \text{const} > 0. \quad (3)$$

By $c > 0$ various constants are denoted.

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Our approach is based on the author's results on the nonlinear differential inequality formulated in Theorem 1 (see [2]–[5]). These results have been used by the author in a study of stability of solutions to abstract nonlinear evolution problems ([5]).

Denote $\mathbb{R}_+ = [0, \infty)$.

Theorem 1. *Let $g \geq 0$ solve the inequality*

$$g'(t) \leq -ag(t) + \alpha(t, g) + \beta(t), \quad t \geq 0, \quad a = \text{const} > 0, \quad (4)$$

where $\alpha(t, g) \geq 0$ and $\beta(t) \geq 0$ are continuous functions of t , $t \in \mathbb{R}_+$ and $\alpha(t, g)$ is locally Lipschitz with respect to g . If there exists a function $\mu(t) > 0$, defined on \mathbb{R}_+ , $\mu \in C^1(\mathbb{R}_+)$, such that

$$\alpha(t, \frac{1}{\mu(t)}) + \beta(t) \leq \frac{1}{\mu(t)} \left(a - \frac{\mu'(t)}{\mu(t)} \right), \quad \forall t \geq 0, \quad (5)$$

and

$$g(0)\mu(0) \leq 1, \quad (6)$$

then g exists on \mathbb{R}_+ and

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad \forall t \geq 0. \quad (7)$$

A proof of Theorem 1 can be found in [5]. Its idea is described in Section 2.

The result of this paper is formulated in Theorem 2.

Theorem 2. *Assume that (2) and (3) hold, $a \geq 2$, $b \geq 2$, $c \in (0, 0.75)$, $p \in (0, \min(0.75a, a_1))$, $R = (b - 1)^{1/b}$. Then any solution to (1) exists on \mathbb{R}_+ and satisfies the estimate*

$$|u(t)| \leq R^{-1}e^{-pt}, \quad \forall t \geq 0, \quad p \in (0, \min(0.25a_1, a)). \quad (8)$$

In Section 2 Theorem 2 is proved.

2 Proof of Theorem 2

Let us reduce equation (1) to the form suitable for an application of Theorem 1. Differentiate (1) and get

$$u' = f' - a \int_0^t e^{-a(t-s)} h(u(s)) ds + h(u(t)). \quad (9)$$

Let $g(t) := |u(t)|$ and take into account that $|F(t)| \leq ce^{a_1 t}$.

From (1) one gets $\int_0^t e^{-a(t-s)} h(u(s)) ds = u - f$. This and equation (9) imply $u' = f' - a(u - f) + h(u(t))$. Therefore, one gets

$$u' = -au + h(u) + F, \quad F := f' + af \quad (10)$$

Multiply (10) by \bar{u} , where \bar{u} stands for complex conjugate of u , and get

$$u'\bar{u} = -ag^2 + h(u)\bar{u} + F\bar{u}. \quad (11)$$

One has

$$u'\bar{u} + u(\bar{u})' = \frac{dg^2}{dt} = 2gg'. \quad (12)$$

We define the derivative as $g' = \lim_{h \rightarrow +0} \frac{g(t+h) - g(t)}{h}$. With this definition, $g(t)$ is differentiable at every point if $u(t)$ is continuously differentiable for all $t \geq 0$. Any solution $u(t)$ to (1) is continuously differentiable under our assumptions. Take complex conjugate of (11), add the resulting equation to (11) and take into account (12). This yields

$$2gg' = -2ag^2 + 2\operatorname{Re}(h(u)\bar{u}) + 2\operatorname{Re}(F\bar{u}). \quad (13)$$

Since $g \geq 0$, one derives from (13), using assumptions (2) and (3), that

$$g'(t) \leq -ag(t) + cg^b + ce^{-a_1 t}. \quad (14)$$

Let

$$\mu(t) = Re^{pt}, \quad R = \text{const} > 0, \quad p \in (0, \min(0.25a, a_1)). \quad (15)$$

Condition (5) can be written as

$$\frac{c}{R^b e^{bpt}} + ce^{-a_1 t} \leq \frac{1}{Re^{pt}}(a - p), \quad t \in \mathbb{R}_+. \quad (16)$$

This inequality holds if

$$\frac{c}{R^{b-1} e^{(b-1)pt}} + cRe^{-(a_1 - p)t} \leq \frac{3a}{4}, \quad t \in \mathbb{R}_+. \quad (17)$$

Inequality (17) holds if

$$\frac{1}{R^{b-1}} + R \leq \frac{3a}{4c}. \quad (18)$$

The minimum of the left side of (18) is attained at $R = (b-1)^{1/b}$ and is equal to $\frac{b}{(b-1)^{(b-1)/b}}$. Thus, (18) holds if

$$\frac{b}{(b-1)^{(b-1)/b}} \leq \frac{3a}{4c}. \quad (19)$$

For example, assume that

$$a \geq 2, \quad c \leq 0.75.$$

Then (19) holds if $b \leq 2(b-1)^{(b-1)/b}$, that is, if

$$b^b \leq 2^b(b-1)^{b-1}. \quad (20)$$

Inequality (20) holds if $b \geq 2$. Thus, by Theorem 1, any solution $u(t)$ of (1) exists globally and

$$|u(t)| \leq \frac{e^{-pt}}{R}, \quad (21)$$

provided that

$$|u(0)|R \leq 1, \quad R = (b-1)^{1/b}, \quad a \geq 2, \quad b \geq 2, \quad c = 0.75, \quad p \in (0, \min(0.25a, a_1)). \quad (22)$$

Inequality $|u(0)|R \leq 1$ holds if $f(0)R \leq 1$. By assumption (3) this inequality holds if $c \leq \frac{1}{R}$. Theorem 2 is proved. \square

Let us prove existence of a solution to (1) using the contraction mapping principle and Theorem 2.

By estimate (21) one has $|u(t)| \leq \frac{1}{R}$ for all $t \geq 0$. Therefore, using assumptions (2) and (3), one gets

$$|Tu| \leq c + \frac{c}{aR^b} \leq \frac{1}{R}, \quad (23)$$

provided that $cR \leq \frac{1}{1 + \frac{1}{aR^b}}$. For $R = (b-1)^{1/b}$ this inequality holds if c is sufficiently small. If (23) holds, then T maps the ball $B_R := \{u : \|u\| \leq \frac{1}{R}\}$ into itself. Here $\|u\| = \max_{t \geq 0} |u(t)|$.

On the ball B_R the operator T is a contraction:

$$\|Tu - Tv\| \leq \left\| \int_0^t e^{-a(t-s)} c |\eta|^{b-1} |ds| \right\| \|u - v\| \leq \frac{c}{R^{b-1}a} \|u - v\|, \quad (24)$$

where the assumption (2) was used, and η is the "middle" element between u and v , $\|\eta\| \leq \frac{1}{R}$. The integral in (24) is estimated as follows:

$$\left\| \int_0^t e^{-a(t-s)} c |\eta|^{b-1} |ds| \right\| \leq \frac{c}{R^{b-1}} \max_{t \geq 0} \int_0^t e^{-a(t-s)} ds \leq \frac{c}{R^{b-1}a} \quad (25)$$

If

$$\frac{c}{R^{b-1}a} < 1, \quad (26)$$

then T is a contraction on B_R . Condition (26) holds if c is sufficiently small. Thus, if condition (26) and the assumptions of Theorem 2 hold, then, by the contraction mapping principle, there exists a unique solution to (1) in the ball B_R . \square

For convenience of the reader we sketch the idea of the proof of Theorem 1 following [2]–[5].

Inequality (5) can be written for the function $w = \frac{1}{\mu}$ as follows:

$$-aw + \alpha(t, w) + \beta(t) \leq w'. \quad (27)$$

From (4) and (27) by a comparison lemma for ordinary differential equations it follows that

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \quad (28)$$

provided that $g(0) \leq w(0) = \frac{1}{\mu(0)}$. The last inequality is the assumption (6). Since $\mu(t) > 0$ and is assumed to be defined for all $t \geq 0$, the function $w = \frac{1}{\mu}$ is defined for all $t \geq 0$. Since $0 \leq g(t) \leq \frac{1}{\mu(t)}$, and $g(t) := |u(t)|$, the function u is defined for all $t \geq 0$.

If $\lim_{t \rightarrow \infty} \mu(t) = 0$, then $\lim_{t \rightarrow \infty} |u(t)| = 0$ by estimate (28).

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